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Incompressible flow along a corner

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The incompressible viscous flow along a right-angle corner, formed by the intersection of two semi-infinite flat plates, is considered. The effect of the three-dimensional geometry on the second-order 'boundary layer' flow away from the corner is determined and an interesting secondary flow is deduced. It is observed that this cross-flow prescribes the necessary asymptotic boundary conditions for the equations governing the flow inside the 'corner layer'. A systematic matching scheme is specified and the corner flow problem is reformulated in terms of the 'corner layer–boundary layer' matching conditions.

1. Introduction

The viscous flow along one of the corners that is formed by the intersection of two semi-infinite perpendicular flat plates (figure 1) is typical of problems of the boundary-region variety. Unlike high-Reynolds-number viscous flows over bodies with small surface curvature, for which two-dimensional boundary-layer theory is generally applicable, these geometries are inherently three-dimensional. The coupling that is created by the mutual interaction of the boundary-layer motion at different points across the surface makes these configurations difficult to analyse.

The corner geometry was first investigated by Carrier (1947). His solution involved a rather arbitrary split of the continuity equation which has been criticized by Kemp (1951) and others, as the cross-plane vorticity equation remained unsatisfied. In turn, Dowdell (1952) attempted to estimate the error that was incurred, by linearizing the full boundary-layer equations about the Carrier solution.

The subsequent literature contains approximate integral solutions by Loitsianskii & Bolshakov (1951) and Bloom & Rubin (1961) for compressible as well as incompressible flow, a series-expansion technique by Levy (1959) and an unsteady Rayleigh-type analogy by Sowerby & Cooke (1953). However, the basic three-dimensionality of such a configuration has not been considered. More recently Howarth & Stewartson (1960) and Stewartson (1961), when discussing the viscous flow over a quarter-infinite flat plate, pointed out the fundamental significance of the three-dimensional first-order potential flow when dealing with these geometries.

It is the purpose of this paper to reformulate the problem of flow along a corner as a singular perturbation problem. The proper asymptotic boundary conditions for the flow in the 'corner-layer' (region IV of figure 1) are determined, and the velocity and pressure distributions within the three-dimensional boundary layers away from the corner (regions II and III of figure 1) are discussed.

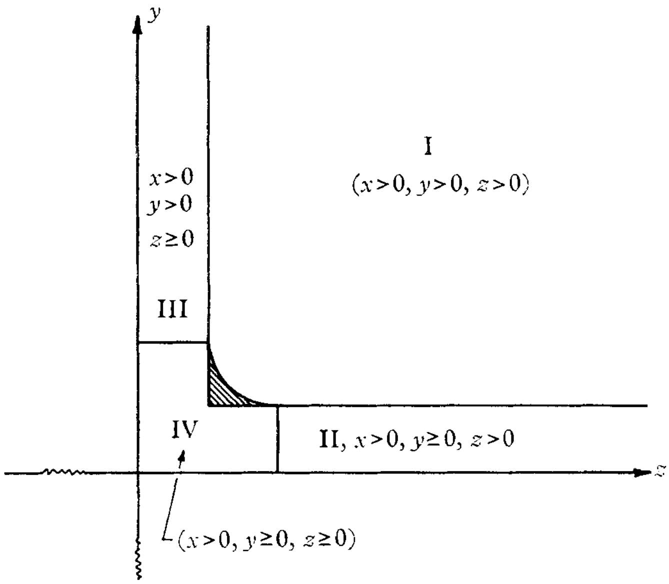

FIGURE 1. Corner flow geometry.

2. Analysis

The incompressible boundary layer in a corner can be suitably represented by four distinct regions, which are depicted in figure 1. Section I is denoted as the potential flow, since to the orders considered here, the effects of rotationality are zero and since the inviscid Laplace equation is applicable there. Strips II and III are designated as the boundary layers and are distinguished by stipulating the plane of interest: $z = 0$, $y > 0$ or $y = 0$, $z > 0$. In these layers the usual two-dimensional boundary-layer equations apply to first-order, with three-dimensional effects, a result of the geometry and subsequent first-order potential flow, appearing in the higher-order equations.

This concept, which was first expressed by Howarth & Stewartson† (1960) and Stewartson (1961) prescribes a precise matching scheme in order to completely specify the boundary-layer motion to any desired order. More significant is the conclusion that the flow in region IV, which is termed the 'corner layer', can only be evaluated by this systematic matching procedure. The equations that govern the flow in the different sections are coupled in the typical boundary-layer sense, by virtue of the boundary conditions; e.g. it is necessary to require that asymptotically the contiguous layers must merge smoothly.

The resulting matching problem is of the following form:

(i) From the known zeroth-order potential flow it is possible to evaluate the first-order boundary-layer solution. This is the Blasius solution for the flat surfaces considered here.

† See footnote † on page 11.

(ii) The first-order potential flow is determined by considering Laplace's equation in the corner region bounded by the planes $y = 0$, $z > 0$ and $z = 0$, $y > 0$. It is at this juncture that the three-dimensional geometry or mutual interaction first appears as a factor in the problem. As a result of this interaction, a cross-flow component of velocity is induced and it is of the same order as the normal velocity issuing from the boundary layer; e.g.

$$\lim_{y\to 0,\, z>0} w(x,y,z) = O(v \text{ in the boundary layer}).$$

(iii) This cross-flow induces a second-order boundary-layer motion that is not present in ordinary two-dimensional theory. At any finite distance from the line of intersection the boundary layer is three-dimensional and only asymptotically becomes two-dimensional at very large distances along the wall.

This entire concept is fundamental to any three-dimensional boundary-layer problem of the boundary-region variety.

In order that the solutions in the various regions be valid at the leading edge ($x = 0$) an additional layer represented by some characteristic dimension would have to be defined there. Since the evaluation is omitted, all solutions are valid only at distances $x > x_0 > 0$, where x_0 is unspecified, although it is expected to be rather small.

3. First-order boundary layer

In order to distinguish between the several regions the following convention is prescribed. Potential-flow variables are denoted by upper-case letters, boundary-layer quantities by lower-case letters, and corner-layer properties by asterisks.

The complete non-dimensionalized Navier–Stokes equations, for an arbitrary region, are of the form

$$(\bar{\mathbf{q}}\,.\,\bar{\nabla})\,\bar{\mathbf{q}} + \bar{\nabla}\bar{p} = (1/Re)\,\bar{\nabla}^2\bar{\mathbf{q}}, \tag{1}$$

and the equation of mass continuity is

$$\bar{\nabla}\,.\,\bar{\mathbf{q}} = 0. \tag{2}$$

All velocities have been non-dimensionalized with respect to the known zeroth-order potential velocity U, all lengths with respect to L, some characteristic dimension, and the pressure with twice the dynamic pressure $\frac{1}{2}\rho U^2$.

The zeroth-order potential flow ($u = U$, $v = w = 0$) is known to satisfy equations (1) and (2) exactly, as well as all the boundary conditions, save that $u = 0$ at the surface. In order to satisfy this additional constraint the concept of a thin shear layer or boundary layer was proposed by Prandtl, who demonstrated the necessity for a layer of thickness of order $Re^{-\frac{1}{2}}$. More recently the procedure has been formalized, by Kaplun (1954), and Van Dyke (1964) among others, with a systematic matching procedure. This apporoach is applicable to the present study and therefore the following series expansions are assumed:

(i) Potential flow:

$$u = U_0 + \epsilon_1 U_1 + \ldots +, \quad v = \epsilon_1 V_1 + \epsilon_2 V_2 + \ldots +,$$
$$w = \epsilon_1 W_1 + \epsilon_2 W_2 + \ldots, \quad p = P_0 + \epsilon_1 P_1 + \epsilon_2 P_2 + \ldots. \tag{3}$$

(ii) Boundary layer:

$$u = u_0 + \delta_1 u_1 + \ldots, \qquad v = \delta_1 v_1 + \delta_2 v_2 + \ldots,$$
$$w = \delta_1 w_1 + \delta_2 w_2 + \ldots, \quad p = p_0 + \delta_1 p_1 + \delta_2 p_2 + \ldots, \tag{4}$$

(iii) Corner layer:

$$u = u_0^* + \gamma_1 u_1^* + \ldots +, \qquad v = \gamma_1 v_1^* + \gamma_2 v_2^* + \ldots,$$
$$w = \gamma_1 w_1^* + \gamma_2 w_2^* + \ldots, \quad p = p_0^* + \gamma_1 p_1^* + \gamma_2 p_2^* + \ldots. \tag{5}$$

The parameters ϵ_i, δ_i and γ_i are constants chosen such that $\epsilon_{i+1} \ll \epsilon_i \ll 1$ and similarly for δ_i and γ_i. In general, they are functions of the Reynolds number. All flow variables in all regions are functions of the co-ordinates (x, y, z).

The equations governing the first-order boundary layer are obtained by placing series (4) into the general Navier–Stokes equations and retaining first-order terms in δ_1 only, so that the additional no-slip condition neglected in the potential flow is satisfied. In order to satisfy the continuity equation as well, the following conditions are prescribed:

(1) δ_1 is set equal to $Re^{-\frac{1}{2}}$.

(2) New boundary-layer co-ordinates are defined, such that $\bar{x}$ and $\bar{z}$ are unchanged, but a new co-ordinate $\bar{Y} = \bar{y}\, Re^{\frac{1}{2}}$ is necessary to make all dimensions of order unity, and in this manner the thickness of the boundary layer is defined.

The details of the analysis are omitted, as they are identical in principle to those for the two-dimensional analysis and can be found elsewhere (Van Dyke 1964). It is important to note that for the corner flow there is no characteristic length, other than the distance from the corner intersection line or leading edge and therefore the length L, which was introduced in a strictly formal manner, will cancel in any final result. For the boundary layer II the classical two-dimensional equations are obtained to first-order:

$$\left.\begin{aligned} &\bar{u}_0 \frac{\partial \bar{u}_0}{\partial \bar{x}} + \bar{v}_1 \frac{\partial \bar{u}_0}{\partial \bar{Y}} = \frac{\partial^2 \bar{u}_0}{\partial \bar{Y}^2}, \\ &\frac{\partial p_0}{\partial \bar{Y}} = 0, \quad \frac{\partial \bar{u}_0}{\partial \bar{x}} + \frac{\partial \bar{v}_1}{\partial \bar{Y}} = 0, \end{aligned}\right\} \tag{6}$$

the appropriate boundary conditions being, $\bar{u}_0 = \bar{v}_1 = 0$ at the surface $(\bar{Y} = 0)$, and $\bar{u}_0 \to 1$ as the asymptotic condition outside of the layer $(\bar{Y} \to \infty)$. Analogous equations are obtained in region III, with $\bar{w}_1$ replacing $\bar{v}_1$ and $\bar{Z}$ replacing $\bar{Y}$.

The solution of equations (6) is the well-known Blasius calculation

$$u = u_0 = Uf'(\eta), \quad v = \epsilon_1 v_1 = U(\nu/2Ux)^{\frac{1}{2}}\{\eta f'(\eta) - f(\eta)\}, \tag{7}$$

where $\eta = y(U/2\nu x)^{\frac{1}{2}}$.

This solution exhibits a normal or outflow velocity v which is absent in the zeroth-order potential flow. This velocity, as well as w on the opposing surface, must serve as the matching condition for the next-order potential flow. Where the asymptotic value of v is the classical result,

$$v \to U(\nu/2Ux)^{\frac{1}{2}}\beta \quad \text{as} \quad \eta \to \infty \quad \text{and} \quad \beta \sim 1{\cdot}21678.$$

4. Potential flow

The equations that govern the first-order potential flow are obtained by retaining only those terms that are first-order in ϵ_1 when the assumed expansion (3) is substituted into the general equations (1). The matching condition on the outflow velocities from the respective boundary layers in II and III forces the choice of $Re^{-\frac{1}{2}}$ for ϵ_1. The potential flow co-ordinates remain, of course, the unstretched system (x, y, z).

The governing equations are

$$\left.\begin{aligned} &\rho U\, \partial U_1/\partial x = -\partial P_1/\partial x, \\ \text{or} \quad &P_1 + \rho U U_1 = \text{const.}, \\ \text{and} \quad &\nabla^2 \Phi = 0, \end{aligned}\right\} \tag{8}$$

where Φ is the velocity potential referred to velocities $(\epsilon_1 U_1, \epsilon_1 V_1, \epsilon_1 W_1)$; e.g. $\partial\Phi/\partial x = \epsilon_1 U_1$, etc. The first-order potential flow remains irrotational.

The Laplace operator ∇^2 is three-dimensional, and equation (8) must be solved within the volume bounded by the planes $y = 0$, $z > 0$ and $z = 0$, $y > 0$. The necessary boundary conditions are a result of the matching conditions

$$\left.\begin{aligned} \lim_{\substack{Y\to\infty \\ y \text{ fixed}}} \delta_1 v_1(x, Y, z) &= \lim_{\substack{y\to 0 \\ Y \text{ fixed}}} \epsilon_1 V_1(x, y, z) = \beta U(\nu/2Ux)^{\frac{1}{2}}, \\ \lim_{\substack{Z\to\infty \\ z \text{ fixed}}} \delta_1 w_1(x, y, Z) &= \lim_{\substack{z\to 0 \\ Z \text{ fixed}}} \epsilon_1 W_1(x, y, z) = \beta U(\nu/2Ux)^{\frac{1}{2}}. \end{aligned}\right\} \tag{9}$$

The boundary conditions for v and w are evaluated directly on $y = 0$ and $z = 0$ respectively, since this represents the first term in a Taylor series expansion and is valid to order $Re^{-\frac{1}{2}}$, the order of the boundary-layer thickness.

The following boundary-value problem is posed: $\nabla^2\Phi = 0$,

$$\left.\begin{aligned} \Phi_y &= \beta U(\nu/2Ux)^{\frac{1}{2}}, \quad y = 0^+, \quad x > 0, \quad z > 0; \\ &\qquad \Phi_y = 0, \quad y = 0^+, \quad x \leqslant 0, \quad z > 0, \\ \Phi_z &= \beta U(\nu/2Ux)^{\frac{1}{2}}, \quad y > 0, \quad x > 0, \quad z = 0^+; \\ &\qquad \Phi_z = 0, \quad y > 0, \quad x \leqslant 0, \quad z = 0^+, \end{aligned}\right\} \tag{10}$$

with the appropriate decay at infinity.

The solution is found directly with the use of a suitable Green's function. We find that,

$$-2\pi\Phi(x, y, z) = \beta(\tfrac{1}{2}\nu U)^{\frac{1}{2}}\{I_1(x, y, z; L_1) + I_2(x, y, z; L_1) + I_3(L_1)\},\dagger \quad \lim L_1 \to \infty, \tag{11}$$

where

$$\left.\begin{aligned} I_1 &= \int_0^{L_1}\int_0^{\infty} dx_0\, dy_0\, x_0^{-\frac{1}{2}}\left[\{(x_0-x)^2 + (y_0-y)^2 + z^2\}^{-\frac{1}{2}} + \{(x_0-x)^2 + (y_0+y)^2 + z^2\}^{-\frac{1}{2}}\right], \\ I_2 &= \int_0^{L_1}\int_0^{\infty} dx_0\, dz_0\, x_0^{-\frac{1}{2}}\left[\{(x_0-x)^2 + y^2 + (z_0-z)^2\}^{-\frac{1}{2}} + \{(x_0-x)^2 + y^2 + (z_0+z)^2\}^{-\frac{1}{2}}\right]. \end{aligned}\right\} \tag{12}$$

† $I_3(L_1)$ is a surface integral over the quarter infinite cylindrical surface $y^2 + z^2 = r^2 = L_1^2$. Since $\nabla\phi \to 0$ as $r \to \infty$, $\nabla I_3(L_1) \to 0$ as $L_1 \to \infty$ and is of no subsequent importance in the analysis for the velocities.

The first-order velocity components are

$$\frac{\partial\Phi}{\partial z} = \epsilon_i W_1 = \frac{\tilde{\zeta}\beta(\frac{1}{2}\nu U)^{\frac{1}{2}}}{\pi x^{\frac{1}{2}}}\int_0^\infty \frac{dt}{t^{\frac{1}{2}}\{(t-1)^2+\tilde{\zeta}^2\}}, \tag{13}$$

$$\frac{\partial\Phi}{\partial x} = \epsilon_1 U_1 = -\frac{\beta(\frac{1}{2}\nu U)^{\frac{1}{2}}}{\pi x^{\frac{1}{2}}}\left\{\int_0^\infty \frac{(t-1)\,dt}{t^{\frac{1}{2}}\{(t-1)^2+\tilde{\eta}^2\}}+\int_0^\infty \frac{(t-1)\,dt}{t^{\frac{1}{2}}\{(t-1)^2+\tilde{\zeta}^2\}}\right\}, \tag{14}$$

$$\frac{\partial\Phi}{\partial y} = \epsilon_1 V_1 = \frac{\tilde{\eta}\beta(\frac{1}{2}\nu U)^{\frac{1}{2}}}{\pi x^{\frac{1}{2}}}\int_0^\infty \frac{dt}{t^{\frac{1}{2}}\{(t-1)^2+\tilde{\eta}^2\}}, \tag{15}$$

where $\tilde{\zeta} = z/x$ and $\tilde{\eta} = y/x$.

Integrals of the type shown above can be easily evaluated for $\tilde{\zeta} > 0$ and $\tilde{\eta} > 0$ by using the complex variable and the proper contour encircling the branch point at $t = 0$.

We find that,

$$\partial\Phi/\partial z = \beta U(\nu/2Ux)^{\frac{1}{2}}(1+\tilde{\zeta}^2)^{-\frac{1}{2}}\operatorname{Re}(1+i\tilde{\zeta})^{\frac{1}{2}}, \tag{13a}$$

$$\partial\Phi/\partial x = -\beta U(\nu/2Ux)^{\frac{1}{2}}\{\tilde{\zeta}^{-1}(1-(1+\tilde{\zeta}^2)^{\frac{1}{2}})\operatorname{Re}(1+i\tilde{\zeta})^{\frac{1}{2}} + \tilde{\eta}^{-1}(1-(1+\tilde{\eta}^2)^{-\frac{1}{2}})\operatorname{Re}(1+i\tilde{\eta})^{\frac{1}{2}}\}, \tag{14a}$$

$$\partial\Phi/\partial y = \beta U(\nu/2Ux)^{\frac{1}{2}}(1+\tilde{\eta}^2)^{-\frac{1}{2}}\operatorname{Re}(1+i\tilde{\eta}), \tag{15a}$$

where $$\sqrt{2}\operatorname{Re}(1+i\tilde{\zeta})^{\frac{1}{2}} = \{1+(1+\tilde{\zeta}^2)^{\frac{1}{2}}\}^{\frac{1}{2}}$$

and similarly for $\operatorname{Re}(1+i\tilde{\eta})$.

Since the asymptotic results will be required in the subsequent analysis, they are quoted below for future reference.

(i) For small† $\tilde{\eta}$ and large $\tilde{\zeta}$

$$\left.\begin{aligned}
\frac{\partial\Phi}{\partial z} &= \frac{\beta U}{\pi}\left(\frac{\nu}{2Uz}\right)^{\frac{1}{2}}\int_0^\infty \frac{d\phi}{\phi^4+1} = v = (\tfrac{1}{2}\beta)(\nu/Uz)^{\frac{1}{2}}U,\\
\frac{\partial\Phi}{\partial x} &= -\frac{\beta U}{\pi}\left(\frac{\nu}{2Uz}\right)^{\frac{1}{2}}\int_0^\infty \frac{\phi^2 d\phi}{\phi^4+1} - \frac{\beta U}{2}\tilde{\eta}\left(\frac{\nu}{2Ux}\right)^{\frac{1}{2}},\\
&= -(\tfrac{1}{2}\beta)U(\nu/Uz)^{\frac{1}{2}} - (\tfrac{1}{2}\beta)\tilde{\eta}U(\nu/2Ux)^{\frac{1}{2}},\\
\partial\Phi/\partial y &= \beta U(\nu/2Ux)^{\frac{1}{2}}\{1+O(\tilde{\eta}^2)\}.
\end{aligned}\right\} \tag{16}$$

(ii) For $\tilde{\eta}$ small† and $\tilde{\zeta}$ small†

$$\left.\begin{aligned}
\partial\Phi/\partial z &= \beta U(\nu/2Ux)^{\frac{1}{2}}\{1-\tfrac{3}{8}\tilde{\zeta}^2+O(\tilde{\zeta}^4)\},\\
\partial\Phi/\partial y &= \beta U(\nu/2Ux)^{\frac{1}{2}}\{1-\tfrac{3}{8}\tilde{\eta}^2+O(\tilde{\eta}^4)\},\\
\partial\Phi/\partial x &= -\tfrac{1}{2}\beta U(\nu/2Ux)^{\frac{1}{2}}(\tilde{\eta}+\tilde{\zeta})\{1+O(\tilde{\eta}^2+\tilde{\zeta}^2)\}.
\end{aligned}\right\} \tag{17}$$

The induced cross-flow, e.g. $\partial\Phi/\partial z$ in the vicinity of the plane $y = 0$, $z > 0$, is of the same order as the out-flow velocity $\partial\Phi/\partial y$, and for small $\tilde{\eta}$ and $\tilde{\zeta}$ they are identical, to order $Re^{-\frac{1}{2}}$. This result is interesting since it implies that, to this order, the effective displacement body would correspond to the simple intersection of

† Since the potential flow (I) is only defined outside the region consisting of the boundary layers and corner layer, the smallness of $\tilde{\eta}$ and $\tilde{\zeta}$ is governed by this restriction. Since the boundary-layer thickness and corner-layer thickness (as will be demonstrated) are of the order $Re^{-\frac{1}{2}}$, the smallness of the second term, on the right-hand side of each of the equations (17), is established.

two parabolic cylinders, which exhibit a right-angle intersection in the cross-plane; i.e. in figure 1 the shaded region is non-existent to order $Re^{-\frac{1}{2}}$. That this is not the case for higher-order solutions is clear from the character of the next-order terms in equation (17).

5. Second-order boundary layer

The induced cross-flow $\partial\Phi/\partial z$ (considering region II with symmetry arguments prescribing the flow in III) leads to a second-order boundary-layer flow which is non-existent in the two-dimensional theory. The pertinent equations are obtained from the second-order terms in the Navier–Stokes expansion of series (4):

$$\bar{u}_0 \partial\bar{u}_1/\partial\bar{x} + \bar{u}_1 \partial\bar{u}_0/\partial\bar{x} + \bar{v}_1 \partial\bar{u}_1/\partial\bar{Y} + \bar{v}_2 \partial\bar{u}_0/\partial\bar{Y} = -\partial\bar{p}_1/\partial\bar{x} + \partial^2\bar{u}_1/\partial\bar{Y}^2, \tag{18}$$

$$\partial\bar{p}_1/\partial\bar{Y} = 0, \tag{19}$$

$$\bar{u}_0 \partial\bar{w}_1/\partial\bar{x} + \bar{v}_1 \partial\bar{w}_1/\partial\bar{Y} = -\partial\bar{p}_1/\partial\bar{z} + \partial^2\bar{w}_1/\partial\bar{Y}^2, \tag{20}$$

$$\partial\bar{u}_1/\partial\bar{x} + \partial\bar{v}_2/\partial\bar{Y} + \partial\bar{w}_1/\partial\bar{z} = 0. \tag{21}$$

From continuity considerations, it has been necessary to choose $\delta_2 = Re^{-1}$. The co-ordinate system (x, Y, z) is defined in § 3.

The z derivative appears only in the continuity equation and therefore all z dependence occurs only parametrically in the unknowns u_1, v_2 and p_1. For the flow near the corner layer, i.e. for small values of the physical co-ordinate z, the w velocity component is independent of z in a first approximation (see equation (17)), and therefore the second term in the expansion, which is proportional to ζ^2, must be used explicitly even in the initial analysis for small values of z.

Although $\nabla\Phi$ has been evaluated in general, the expressions (13) are complex and therefore the second-order boundary-layer distributions are discussed only for very small or very large values of z respectively. The small z solution is needed prior to the corner-layer investigation since the ultimate matching conditions are imposed on u_0, v_1, and w_1 as $z \to 0$.

The pressure p_1 is constant across the layer, and can be determined by considering the asymptotic form of equation (18) or (20) at the outer edge of the boundary layer. Therefore

$$Re^{-\frac{1}{2}}\, \partial\bar{p}_1/\partial\bar{x} = Re^{-\frac{1}{2}}\, \partial\bar{P}_1/\partial\bar{x} = -Re^{-\frac{1}{2}}\, \partial\bar{U}_1/\partial\bar{x}$$
$$= -\partial(\partial\Phi/\partial\bar{x})/\partial\bar{x},$$

or

$$Re^{-\frac{1}{2}} p_e = \text{const.} - \partial\Phi/\partial\bar{x},$$

and

$$\partial\bar{p}_1/\partial\bar{z} = -\partial W_1(\bar{x}, 0, \bar{z})/\partial\bar{x}. \tag{22}$$

The $\bar{w}_1$ distribution is determined directly from equation (20), using the asymptotic condition (22).

It is found that

$$\bar{u}_0 \partial\bar{w}_1/\partial\bar{x} + \bar{v}_1 \partial\bar{w}_1/\partial\bar{Y} = \partial\bar{W}_1(\bar{x}, 0, \bar{z})/\partial\bar{x} + \partial^2\bar{w}_1/\partial\bar{Y}^2. \tag{23}$$

The appropriate boundary conditions are

$$\bar{w}_1 = 0 \quad \text{on} \quad \bar{Y} = 0, \quad \text{and} \quad \bar{w}_1 \to \bar{W}(\bar{x}, 0, \bar{z}) \quad \text{as} \quad \bar{Y} \to \infty.$$

Case I

Small $\tilde{\zeta}$:

$$Re^{-\frac{1}{2}}\overline{W}_1(x,0,z) = \beta(\nu/2Ux)^{\frac{1}{2}}\{1-\tfrac{3}{8}\tilde{\zeta}^2+O(\tilde{\zeta}^4)\}.$$

It is assumed that

$$Re^{-\frac{1}{2}}\overline{w}_1 = \beta(\nu/2Ux)^{\frac{1}{2}}\{H_0'(\eta)-(\tfrac{3}{8}\tilde{\zeta}^2)H_2'(\eta)\}.$$

With $\overline{u}_0$ and $\overline{v}_1$ defined in equation (7), the final equation for $H_0'(\eta)$ becomes

$$H_0'''(\eta)+\{f(\eta)H_0'(\eta)\}' = 1, \tag{24}$$

where

$$H_0'(0) = 0, \quad H_0'(\eta)\to 1 \quad \text{as} \quad \eta\to\infty,$$
$$\eta = y(U/2\nu x)^{\frac{1}{2}}.$$

From equation (24), it is seen that the boundary-layer distribution for w_1, obtained from the first term in the $\tilde{\zeta}$ expansion is identical to the second term $L(\eta)$ for the cross-flow velocity in the 'secondary layer' on a quarter-infinite flat plate (Stewartson 1961). However, for that geometry the leading term in the expansion was logarithmically singular as $z\to 0$.

The solution of (24) is

$$H_0'(\eta) = f''(\eta)\int_0^\eta \frac{t-\beta}{f''(t)}\,dt, \tag{25}$$

and is depicted in figure (2), along with a representation of the Blasius distribution $f'(\eta)$. It is of interest to note that the cross-wise shear component $H_0''(0) = -\beta$, and since $\beta > 0$, $H_0''(0) < 0$. This results in an inward flow (toward the corner) near the surface ($\eta = 0$) which is reversed within the boundary layer until the asymptotic outward flow is reached when $\eta\to\infty$. From the pressure condition (22) we see that the corner layer asymptotes into an adverse pressure gradient, which it apparently cannot overcome near the wall.†

The governing equation for $H_2'(\eta)$ is obtained simply as

$$H_2'''(\eta)+f(\eta)H_2''(\eta)+5f'(\eta)H_2'(\eta) = 5, \tag{26}$$

where

$$H_2'(0) = 0, \quad H_2'(\eta)\to 1 \quad \text{as} \quad \eta\to\infty.$$

If additional terms are to be included in the $W_1(x,0,z)$ expansion then $\overline{w}_1$ can be represented by,

$$Re^{-\frac{1}{2}}\overline{w}_1 = \beta(\nu/2Ux)^{\frac{1}{2}}\sum_{\eta=0}^{\infty}A_{2n}\tilde{\zeta}^{2n}H_{2n}'(\eta),$$

where the A_{2n}'s are constants, obtained directly from the $W_1(x,0,z)$ distribution; e.g. $A_0 = 1$, $A_2 = -\frac{3}{8}$, etc. The functions H_{2n} satisfy equations of the following type:

$$H_{2n}'''(\eta)+f(\eta)H_{2n}''(\eta)+(4n+1)f'(\eta)H_{2n}'(\eta) = 4n+1,\ddagger \tag{27}$$

where $n = 1,2,\ldots$, and $H_{2n}'(0) = 0$, $H_{2n}'(\eta)\to 1$ as $\eta\to\infty$.

† This interpretation is to be found in Pearson's thesis (1957), although it was first suggested to me by Professor Paul A. Libby of the University of California at La Jolla.

‡ See Appendix II.

This expansion is comparable to the Blasius expansion for the flow over a circular cylinder, although it is greatly simplified by the linearity of the governing equation (20). Undoubtedly, other series-expansion techniques, such as the Görtler series (Rosenhead 1963) are applicable as well.

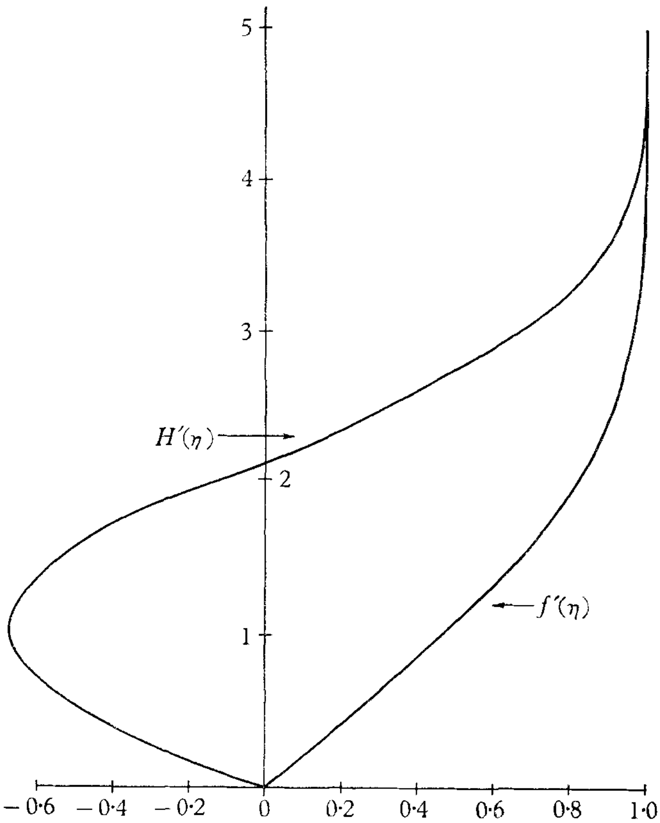

FIGURE 2. Asymptotic cross-flow variation $[H'(\eta)]$ in the boundary layer, and comparison with the Blasius distribution $[f'(\eta)]$.

The solution for $\bar{u}_1$ can be obtained similarly from equation (18), with the continuity equation providing the v_2 boundary-layer distribution. If

$$Re^{-\frac{1}{2}}\bar{u}_1 = -\beta/2(\nu/2Ux)^{\frac{1}{2}}\,\zeta G_1'(\eta),$$

then
$$Re^{-1}\bar{v}_2 = (\beta\tilde{\zeta}/2x)\,(\nu/2Ux)\,\{\hat{H}(\eta)-2G_1(\eta)-\eta G_1'(\eta)\},$$

where
$$\hat{H}(\eta) = \int_0^{\eta} H_2'(\eta)\,d\eta.$$

Therefore,

$$G_1'''(\eta)+(f(\eta)\,G_1'(\eta))'+2\{f'(\eta)\,G_1'(\eta)-f''(\eta)\,G_1(\eta)\} = 3\{1-\hat{H}(\eta)f''(\eta)\}, \quad (28)$$

and
$$G_1(0) = G_1'(0) = 0, \quad G_1'(\eta)\to 1 \quad \text{as} \quad \eta\to\infty.$$

Additional terms in the $\bar{u}_1$ expansion for small $\tilde{\zeta}$ can be determined in a manner similar to that described above for $\bar{w}_1$. The second-order shear stress $(\tau_1)_{xy}$ becomes

$$(\tau_1)_{xy} = -\beta G_1''(0)\,\tilde{\zeta}\,Re_x^{-\frac{1}{2}}(\tfrac{1}{2}\rho U^2)+O(\tilde{\zeta}^3).$$

Therefore the total skin friction is

$$(C_f)_{xy} = 2\tau_{xy}/\rho U^2 = 0{\cdot}664\, Re_x^{-\frac{1}{2}} - \beta G_1''(0)\,\tilde{\zeta}\, Re_x^{-1} + O(\tilde{\zeta}^3 Re_x^{-1}),$$

and from the $\overline{w}_1$ distribution

$$(C_f)_{zy} = -\beta\, Re_x^{-1}\{1 + (\tfrac{3}{8})\,\tilde{\zeta}^2 H_2''(0) + O(\tilde{\zeta}^4)\},$$

where $G_1''(0) = -7{\cdot}0820$, and $H_2''(0) = -8{\cdot}3972$.†

Case II

Large $\tilde{\zeta}$: $$Re^{-\frac{1}{2}} W_1(x,0,z) = 0{\cdot}609(\nu/Uz)^{\frac{1}{2}} = k/z^{\frac{1}{2}},$$

and $$Re^{-\frac{1}{2}}\, \overline{U}(x,0,z) = -k/z^{\frac{1}{2}}.$$

If $Re^{-\frac{1}{2}}\, \overline{W}_1 = (k/z^{\frac{1}{2}})\, h'(\eta)$, then equation (23) becomes

$$h'''(\eta) + f(\eta)\, h''(\eta) = 0,$$

with $$h'(0) = 0, \quad h'(\eta) \to 1 \quad \text{as} \quad \eta \to \infty.$$

It is clear that $h'(\eta) = f'(\eta)$ and therefore the cross-wise velocity decays as $z^{\frac{1}{2}}$ for $\tilde{z} \gg 1$, where it exhibits a Blasius-type profile.

The $\overline{u}_1$ distribution is obtained likewise, by choosing $Re^{-\frac{1}{2}}\,\overline{u}_1 = -(k/z^{\frac{1}{2}})\, g'(\eta)$. From continuity

$$Re^{-1}\,\overline{v}_2 = -(k/z^{\frac{1}{2}})\,(\nu/2Ux)^{\frac{1}{2}}\{\eta g'(\eta) - g(\eta)\}, \tag{29}$$

and therefore $$g'''(\eta) + g''(\eta) f(\eta) + g(\eta) f''(\eta) = 0, \tag{30}$$

with $$g(0) = g'(0) = 0, \quad g'(\eta) \to 1 \quad \text{as} \quad \eta \to \infty.$$

It is seen that $$2g(\eta) = f(\eta) + \eta f'(\eta), \tag{31}$$

and $$Re^{-\frac{1}{2}}\,\overline{u}_1 = -0{\cdot}3045(\nu/Uz)^{\frac{1}{2}}\{\eta f''(\eta) + 2f'(\eta)\}. \tag{32}$$

The second-order shear stress becomes

$$(\tau_1)_{xy} = -0{\cdot}303\, U\mu (xz)^{-\frac{1}{2}}$$

and the total skin-friction coefficient for $\tilde{\zeta} \gg 1$ is

$$(C_f)_{xy} = \tau_{xy}/\tfrac{1}{2}\rho U^2 = 0{\cdot}664\, Re_x^{-\frac{1}{2}} - 0{\cdot}606\, Re_x^{-1}(x/z)^{\frac{1}{2}} + \ldots + . \tag{33}$$

Finally, from the w_1 distribution,

$$(C_f)_{zy} = 0{\cdot}404\, Re_x^{-1}(x/z)^{-\frac{1}{2}} + \ldots + . \tag{34}$$

It is noted that a momentum integral approach is applicable to estimate the skin friction at other points in the boundary layer and in the asymptotic case of $\tilde{\zeta} \to \infty$, it was found to give very good agreement with the exact solution, when the simplest profile was introduced. However, the general solution involves a partial differential equation for the boundary-layer-thickness parameter and therefore its usefulness is in question.

6. Corner layer

The equations governing the behaviour of the motion in the corner layer are obtained by retaining only first-order terms in the Navier–Stokes expansion of series (5). For this analysis, it is necessary to satisfy the no-slip condition on both

† See Appendix II.

surfaces $y = 0$ and $z = 0$, and a sufficient number of terms must therefore be retained in the first-order expansion. Since $\overline{w}_1$ and $\overline{v}_1$ are both of order $Re^{-\frac{1}{2}}$, it is clear from continuity arguments as well, that

(i) $\gamma_1 = Re^{-\frac{1}{2}}$;

(ii) the corner-layer co-ordinates must be stretched such that x is unchanged, but $y = Y\,Re^{-\frac{1}{2}}$ and $z = Z\,Re^{-\frac{1}{2}}$.

With these requirements, the following equations for the corner layer are obtained to first-order:

$$\left.\begin{aligned}
\overline{u}_0^*\,\partial\overline{u}_0^*/\partial\overline{x}+\overline{v}_1^*\,\partial\overline{u}_0^*/\partial\overline{Y}+w_1^*\,\partial\overline{u}_0^*/\partial\overline{Z} &= \partial^2\overline{u}_0^*/\partial\overline{Y}^2+\partial^2\overline{u}_0^*/\partial\overline{Z}^2,\\
\overline{u}_0^*\,\partial\overline{v}_1^*/\partial\overline{x}+\overline{v}_1^*\,\partial\overline{v}_1^*/\partial\overline{Y}+\overline{w}_1^*\,\partial\overline{v}_1^*/\partial\overline{Z} &= -\partial\overline{p}_2^*/\partial\overline{Y}+\partial^2\overline{v}_1^*/\partial\overline{Y}^2+\partial^2\overline{v}_1^*/\partial\overline{Z}^2,\\
\overline{u}_0^*\,\partial\overline{w}_1^*/\partial\overline{x}+\overline{v}_1^*\,\partial\overline{w}_1^*/\partial\overline{Y}+\overline{w}_1^*\,\partial\overline{w}_1^*/\partial\overline{Z} &= -\partial\overline{p}_2^*/\partial\overline{Z}+\partial^2\overline{w}_1^*/\partial\overline{Y}^2+\partial^2\overline{w}_1^*/\partial\overline{Z}^2,\\
\partial\overline{u}_0^*/\partial\overline{x}+\partial\overline{v}_1^*/\partial\overline{Y}+\partial\overline{w}_1^*/\partial\overline{Z} &= 0, \quad \overline{p}_0^* = \text{const.},\\
\partial\overline{p}_1^*/\partial\overline{Y} = 0, &\quad \partial\overline{p}_1^*/\partial\overline{Z} = 0.
\end{aligned}\right\} \qquad (35)$$

In slightly different forms these equations have appeared in several publications and Carrier (1947) has solved them by assuming that a *single* 'potential' function satisfies the continuity equation instead of the usual two stream functions required in three-dimensional calculations. He accomplished this by splitting the continuity equation and disregarding the cross-plane vorticity equation which remained unsatisfied. The magnitude of the error that this assumption imposes was not known precisely; Dowdell (1952) in his master's thesis attempted to evaluate it by linearizing the equations about Carrier's system.

It is now possible to show that the second 'potential' function is not necessarily small and that the boundary conditions which are necessary in order to solve the above system of equations depend closely on the second-order boundary-layer solution.†

In support of the former statement, it suffices to describe the corner-layer behaviour as $Z \to \infty$ with Y fixed. The proper boundary condition on this limit is determined by the boundary-layer result as $z \to 0$. This is

$$w \to \beta U(\nu/2Ux)^{\frac{1}{2}}\,H_0'(\eta).‡$$

The asymptotic result that is obtained by assuming a single 'potential' function is

$$w \sim \beta U(\nu/2Ux)^{\frac{1}{2}} f'(\eta).$$

† It should be noted that the Carrier approach was aimed primarily at determining the mainstream behaviour and may have given reasonable estimates of it, especially near the boundaries of the corner layer. The cross-flow velocities would not be correct.

‡ After this paper was submitted, the Editor drew attention to the existence of two Ph.D. theses, by Dr J. R. A. Pearson (1957) and Dr G. F. Louis (1957), presented at Cambridge University. Both were concerned with the corner-flow problem. Dr Pearson was aware of the necessary matching requirements for the corner-layer solution, and by considering the asymptotic form of the cross-plane vorticity equation, he deduced the correct asymptotic value for w, without a specific knowledge of the entire second-order potential flow. Therefore, it is concluded that this boundary condition should be correct even for the intersection of two quarter-infinite plates. He carried out a relaxation solution that was unsatisfactory, as large errors occurred in the calculation of the cross-flow velocities. The axial-velocity calculation was somewhat better and indicated that Carrier's solution might even be in error for the mainstream velocities. Dr Louis's thesis was unavailable.

Since $f'(\eta) \neq H_0'(\eta)$, and in fact they are considerably different (see figure 2), the single 'potential' assumption does not appear to be justified.

With reference to the second statement, it has been shown that the asymptotic corner-layer conditions are *all pre-determined* and not a consequence of the boundary-layer solution, as is the normal outflow in two-dimensional theory. These values form a complete set of necessary boundary conditions for the corner-layer equations.

The final equations for the corner layer, in terms of the similarity variables η and ζ, are the following:

$$\left.\begin{aligned} -\eta\hat{u}_\eta - \zeta\hat{u}_\zeta + \hat{v}_\eta + \hat{w}_\zeta &= 0; \\ -\eta\hat{u}\hat{u}_\eta - \zeta\hat{u}\hat{u}_\zeta + \hat{v}\hat{u}_\eta + \hat{w}\hat{u}_\zeta &= \hat{u}_{\eta\eta} + \hat{u}_{\zeta\zeta}; \\ -\hat{u}\hat{v} - \eta\hat{u}\hat{v}_\eta - \zeta\hat{u}\hat{v}_\zeta + \hat{v}\hat{v}_\eta + \hat{w}\hat{v}_\zeta &= -\hat{p}_\eta + \hat{v}_{\eta\eta} + \hat{v}_{\zeta\zeta}; \\ -\hat{u}\hat{w} - \eta\hat{u}\hat{w}_\eta - \zeta\hat{u}\hat{w}_\zeta + \hat{v}\hat{w}_\eta + \hat{w}\hat{w}_\zeta &= -\hat{p}_\zeta + \hat{w}_{\eta\eta} + \hat{w}_{\zeta\zeta}; \end{aligned}\right\} \quad (36)$$

where $\overline{u}_1^* = \hat{u}, \quad Re^{-\frac{1}{2}}v_1^* = (\nu/2Ux)^{\frac{1}{2}}\hat{v}, \quad Re^{-\frac{1}{2}}\overline{w}_1^* = (\nu/2Ux)^{\frac{1}{2}}\hat{w},$

$$Re^{-1}\overline{p}_2 = (\nu/2Ux)\hat{p}, \quad \eta = y(U/2\nu x)^{\frac{1}{2}}, \quad \zeta = z(U/2\nu x)^{\frac{1}{2}}.$$

The appropriate boundary conditions are

$$\left.\begin{aligned} &\hat{u} = \hat{v} = \hat{w} = 0 \quad \text{on} \quad \eta = 0 \quad \text{and on} \quad \zeta = 0, \\ &\hat{u} \to f'(\eta), \quad \hat{v} \to \eta f'(\eta) - f(\eta), \quad \hat{w} \to \beta H_0'(\eta) \quad \text{as} \quad \zeta \to \infty, \\ &\hat{u} \to f'(\zeta), \quad \hat{v} \to \beta H_0'(\zeta), \quad \hat{w} \to \zeta f'(\zeta) - f(\zeta) \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \quad (37)$$

In addition,

$$\hat{p}_\eta \to \beta, \quad \hat{w}_\eta \to 0 \quad \text{as} \quad \eta \to \infty, \quad \hat{p}_\zeta \to \beta, \quad \hat{v}_\zeta \to 0 \quad \text{as} \quad \zeta \to \infty.\dagger$$

The solution of these equations (36) with the boundary conditions (37) is quite complex, requiring an elaborate relaxation procedure on a digital computer and work along these lines has already been started.

7. Concluding remarks

The boundary layer in a corner is typical of boundary-layer flows over bodies with large or infinite surface curvature. Several distinct regions are defined and the equations governing the various motions are coupled through the asymptotic boundary conditions. The solution for the three-dimensional boundary layers away from the corner region has been discussed to order $Re^{-\frac{1}{2}}$; the cross-wise velocity distribution has been shown to exhibit an interesting inward flow pattern near the corner that is analogous to a similar motion found for the quarter-infinite flat plate. It has been shown that a series-expansion method can be used to examine the boundary-layer flow at other points along the surface.

The equations and appropriate boundary conditions for the flow in the corner layer have been determined as a result of the consistent matching with the boundary layers, and a numerical solution is being developed.

† From an approximate asymptotic analysis of the corner-layer equations (36) it can be predicted that these conditions, as well as (37), are approached exponentially, and not algebraically, fast. The numerical calculations of Carrier (1947) and Pearson (1957) concur with this conclusion.

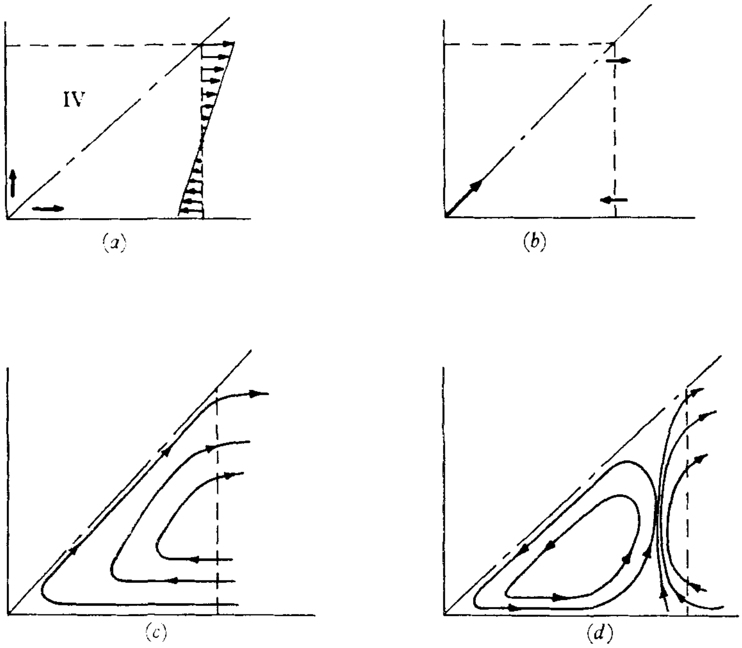

FIGURE 3.

The procedure should be applicable to other three-dimensional geometries, compressible flows and flows over curved surfaces, i.e. for pressure-gradient flows. These possibilities are currently being investigated.

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Appendix I

The asymptotic analysis of § 5 predicts an unusual inflow near the wall (figure 3 (*a*)). From continuity considerations in the immediate vicinity of the line of intersection, an outflow is anticipated and would be of the type shown in figure 3 (*a*) or (*b*). With these asymptotic conditions, it is possible to speculate on the nature of the flow inside the 'corner-layer'.

Two possible cross-flow patterns, which are consistent with the above asymptotic conditions, are shown in figure 3 (*c*) and (*d*). The latter depicts closed streamlines when viewed in the cross-plane, and is a natural result of the opposing velocities in the two asymptotic limits. The former corresponds to inner conditions as shown in sketch (*b*).

Some recent turbulent corner-flow experiments (Gessner & Jones 1961; Paradis 1963) show flow patterns similar to the speculation of sketch (*d*). While these results might have been the effect of turbulent fluctuations or the experimental geometry, they are consistent with the analyses described in this paper.

Appendix II

In determining the solutions of the equations governing the second-order boundary layer (equations (27), (28), etc.), it has been necessary to allow only exponentially-decaying solutions. Discussion on this point is provided by Stewartson (1957), and Van Dyke (1964) among others.

By examining the asymptotic form of equations (27), it can be shown that if $H'_{2n}(\eta) = 1+e^{-\frac{1}{4}\eta^2}P_{2n}(\eta)$, then $P_{2n}(\eta)$ satisfies Weber's equation (Whittaker & Watson 1962),

$$P''_{2n}(\eta)+\{(4n+\tfrac{1}{2})-\tfrac{1}{4}\eta^2\}P_{2n}(\eta) = 0,$$

whereby the asymptotic form of $H'_{2n}(\eta)$ is described by an exponentially decaying solution of $O(\eta^{4n}e^{-\frac{1}{2}\eta^2})$ and an algebraically decaying solution of $O(\eta^{-4n-1})$. Therefore, in the numerical computation, as $H'_{2n}(\eta)\to 1$ for $\eta\to\infty$, it is necessary to filter out the algebraically-decaying part. This becomes increasingly difficult as n becomes larger, but for $n = 1$ it poses no great difficulty. Curiously, for $n = 0$, the requirement that $H''_0(0) = -\beta$ identically eliminates the linear algebraic decay, for then $f(\eta)H'_0(\eta) \sim \eta-\beta$.

Similar arguments hold for $G_1(\eta)$ (which allows for decay of the form η^{-3}) and all subsequent $G_n(\eta)$ solutions.

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